Gamow's Theory of α-decay :-

Gamow's theory is regarded as a successful theory of α-decay due to the following reasons:-

(1). The theory helps to derive an expression for the decay constant λ.

(2). It explains the phenomenon of α-disintegration.

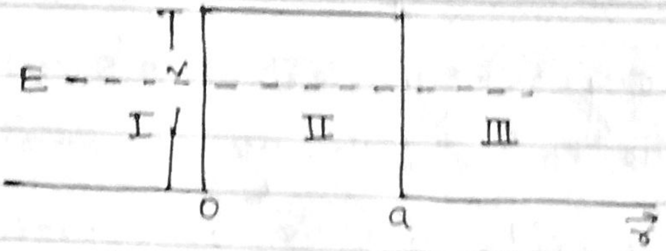
(3). A relation between λ and E of the α-particle similar to Geiger and Nuttall can be established with its help.

If the motion of an α-particle be treated wave mechanically, then it will be found that there is a finite probability that the particle can leak through the barrier even though the KE(E) is less than the height of the barrier(V). The probability that an α-particle can leak through the barrier (tunnel effect) can be calculated as follows:-

Consider the one-dimensional potential barrier is rectangular in shape of width a and hight V ($V>E$). There are three regions of interest, the Scrodinger time independent equation in regions I and III is

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2} E u = 0 \quad \text{---(1)}$$

where $m = \frac{M_\alpha M_D}{(M_\alpha + M_D)}$ is the reduced mass of the α-particle and the nucleus.



Scrodinger equation for region II is

$$\frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}(E-V)u = 0 \quad \text{———} (2)$$

since in region I has both incident and reflected α- waves, the solution of eqn (1) is

$$u_1 = A_1 e^{ik_1 r} + B_1 e^{-ik_1 r} \quad \text{———} (3)$$

$$\text{where } k_1^2 = \frac{2mE}{\hbar^2} \quad \text{———} (4)$$

The region II has both forward moving transmitted wave and reflected wave from the other side of the barrier. so, the solution of eqn (4) is

$$u_2 = A_2 e^{k_2 r} + B_2 e^{-k_2 r} \quad \text{———} (5)$$

$$\text{where } k_2^2 = \frac{2m(V-E)}{\hbar^2} \quad \text{———} (6)$$

The region III has only forward moving transmitted wave as such as solution of (6) is

$$u_3 = A_3 e^{ik_1 r} \quad \text{———} (7)$$

The constants $A_1, A_2, A_3, B_1,$ and B_2 are to be determined from the following Boundary conditions:—

(i) $u_1 = u_2$ and $\frac{\partial u_1}{\partial r} = \frac{\partial u_2}{\partial r}$ at $r=0$

(ii) $u_2 = u_3$ and $\frac{\partial u_2}{\partial r} = \frac{\partial u_3}{\partial r}$, at $r=a$

Substituting the values of u_1, u_2 and u_3 in the above relations, we get

$$A_1 + B_1 = A_2 + B_2 \quad \text{—— (8)}$$

$$iK_1 A_1 - iK_1 B_1 = K_2 A_2 - K_2 B_2 \quad \text{—— (9)}$$

$$A_2 e^{K_2 a} + B_2 e^{-K_2 a} = A_3 e^{iK_1 a} \quad \text{—— (10)}$$

$$A_2 K_2 e^{K_2 a} - B_2 K_2 e^{-K_2 a} = iK_1 A_3 e^{iK_1 a} \quad \text{—— (11)}$$

from eqn (10) & (11), we have

$$A_2 = \frac{1}{2} A_3 \left(1 + \frac{iK_1}{K_2}\right) \cdot e^{i(K_1 - K_2)a} \quad \text{—— (12)}$$

$$\& \; B_2 = \frac{1}{2} A_3 \left(1 - \frac{iK_1}{K_2}\right) \cdot e^{i(K_1 + K_2)a} \quad \text{—— (13)}$$

again from eqn (8) & (9), we have,

$$A_1 = \frac{1}{2} A_2 \left(1 + \frac{K_2}{iK_1}\right) + \frac{1}{2} B_2 \left(1 - \frac{K_2}{iK_1}\right) \quad \text{—— (14)}$$

Substituting the value of A_2 and B_2 in eqn (14) we have

$$A_1 = \frac{1}{4} A_3 \left(1 + \frac{iK_1}{K_2}\right)\left(1 + \frac{K_1}{iK_1}\right) e^{(iK_1 - K_2)a} + \frac{1}{4} A_3 \left(i - \frac{iK_1}{K_2}\right)\left(1 - \frac{K_2}{iK_1}\right) e^{(iK_1 + K_2)a} \quad \text{—— (15)}$$

Now, the velocity of the α-particle in region I is the same as that in region-III, the transmission probability of the incident α-particle is

$$T' = \frac{\text{Incident flux}}{\text{Transmitted flux}} = \frac{|A_1|^2 v}{|A_3|^2 v} = \frac{|A_1|^2}{|A_3|^2} \quad \text{—— (16)}$$

Practically $2K_2a >> 1$, hence the first term (15) can be Neglected in comparision to the second.

$$\therefore T' = \frac{|A_1|^2}{|A_3|^2} = \left(\frac{A_1}{A_3}\right)\left(\frac{A_1}{A_3}\right)^* = \frac{1}{16}\left(1-\frac{iK_1}{K_2}\right)\left(1+\frac{iK_2}{K_2}\right)\left(1-\frac{K_2}{iK_1}\right)\left(1+\frac{K_2}{iK_1}\right) e^{2K_2a}$$

$$= \frac{(K_1^2+K_2^2)^2}{16K_1^2K_2^2} \cdot e^{2K_2a}$$

$\therefore$ Transmittivity of the barrier is

$$\frac{T-1}{T'} = \frac{16K_1^2K_2^2}{(K_1^2+K_2^2)^2} \cdot e^{-2K_2a} \quad \text{—— (17)}$$

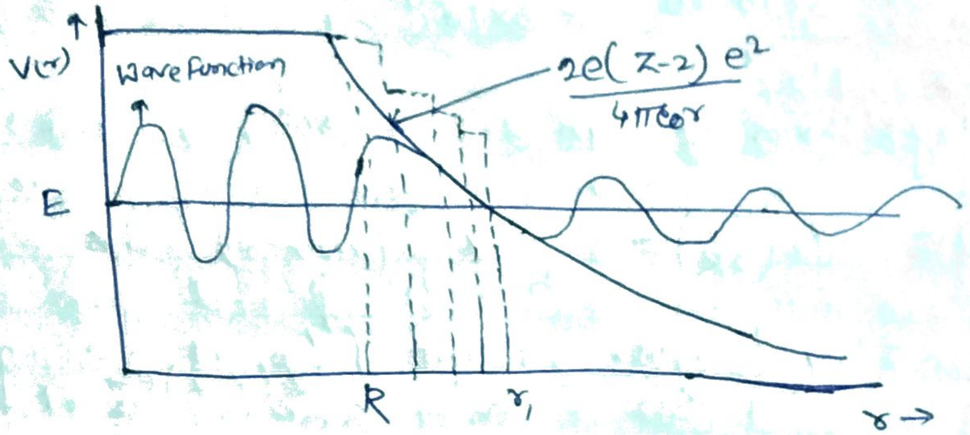
Since, $2K_2a >> 1$, e^{-2K_2a} is extremely small. The factor $\frac{16K_1^2K_2^2}{(K_1^2+K_2^2)^2}$ is usually of the ~~factor~~ order of magnitude unity (the maximum value is four). So for the calculation of the order of magnitude is

$$\boxed{T = e^{-2K_2a}} \quad \text{—— (18)}$$

eqn (18) represents the fraction of the α-particle of the α-particle that can penetrate the barrier of width 'a' and height $V > E$

In case V is not constant in the region $0<r<a$, we can approximate it with a series of small steps each with a constant potential. since, the total probability is the product of the individual probabilities, Assuming the intervals smaller and smaller, the sum goes over to an integration, so that

$$T = e^{-2\int K_2 dr} \quad \text{——— (19)}$$



The integration extends over the region from $r=R$ to $r=r_1$ where $V(r)>E$.

Substituting the value of K_2 in (19), we get

$$T = \exp\left\{-\frac{2\sqrt{2m}}{\hbar}\int_R^{r_1}[V(r)-E]^{1/2}dr\right\} \quad \text{——— (20)}$$

If the α- particle moves inside the potential well with a velocity v_0 and hits the wall $\omega = \frac{v_0}{2R}$ time per second, R being the radius of the nucleus. There is a probability (P) of leaking at each hit of the α-particle.

The rate of emission of α-particle, i.e decay constant λ would then be,

$$\lambda = \omega T = \frac{v_0}{2R}\exp\left\{-\frac{2\sqrt{2m}}{\hbar}\int_R^{r_1}[V(r)-E]^{1/2}dr\right\} \quad \text{—— (21)}$$

Since, the speed of the α-particle is $\sim 3\times10^{6}$ m/s and the diameter of the nucleus $\sim 10^{-14}$ m, the no. of Collisions made by an α-particle against the nucleus wall in one second is

$$\sim \frac{3\times10^{6}\, m}{10^{-14}\, m} = 3\times10^{20}$$

Taking logrithom of eqn (21), we have

$$\log_e \lambda = \log_e\left(\frac{v_0}{2R}\right) - \frac{2\sqrt{2m}}{\hbar}\int_R^{r_1}\left(V(r)-E\right)^{1/2} dr$$

$$= \log_e\left(\frac{v_0}{2R}\right) - \frac{2\sqrt{2mE}}{\hbar}\int_R^{r_1}\left[\frac{2(Z-2)e^2}{4\pi\epsilon_0 r E} - 1\right]^{1/2} dr \quad (22)$$

Since, $E = \dfrac{2(Z-2)e^2}{4\pi\epsilon_0 r_1}$

The upper limit of integral is $r_1 = 2(Z-2)e^2/4\pi\epsilon_0 E$

Substituting $r = r_1\cos^2\psi$ and $R = r_1\cos^2\psi_0$, we have

$$\log_e \lambda = \log_e\left(\frac{v_0}{2R}\right) + \frac{4\sqrt{2mE}\, r_1}{\hbar^2}\int_{\psi_0}^{0}\sin^2\psi\, d\psi$$

$$= \log_e\left(\frac{v_0}{2R}\right) + \frac{2\sqrt{2mE}\, r_1}{\hbar}\left[-\psi_0 + \sin\psi_0\cos\psi_0\right]$$

$$= \log_e\left(\frac{v_0}{2R}\right) - \frac{2\sqrt{2mE}\, r_1}{\hbar^2}\left[\cos^{-1}\left(\frac{R}{r_1}\right)^{1/2} - \left(\frac{R}{r_1}\right)^{1/2}\left(1-\frac{R}{r_1}\right)^{1/2}\right]$$

Since $R \ll r_1$, we may write,

$\cos^{-1}\left(\frac{R}{r_1}\right)^{1/2} \simeq \frac{\pi}{2} - \left(\frac{R}{r_1}\right)^{1/2}$ and $\left(1-\frac{R}{r_1}\right)^{1/2} \simeq 1$

Therefore, $\log_e \lambda = \log_e\left(\frac{v_0}{2R}\right) - \frac{2\sqrt{2mE}}{\hbar} r_1 \left[\frac{\pi}{2} - 2\left(\frac{R}{r_1}\right)^{1/2}\right]$

$$= \log_e\left(\frac{v_0}{2R}\right) + \frac{4e}{\hbar}\left(\frac{m}{\pi\epsilon_0}\right)^{1/2}(z-2)^{1/2} R^{1/2} - \frac{e^2}{\hbar\epsilon_0}\left(\frac{m}{2}\right)^{1/2}(z-2)\, E^{-1/2}$$

$$= \log_e\left(\frac{v_0}{2R}\right) + 2.97\, z_D^{1/2} R^{1/2} - 3.95\, z_D E^{-1/2} \quad \text{--- (23)}$$

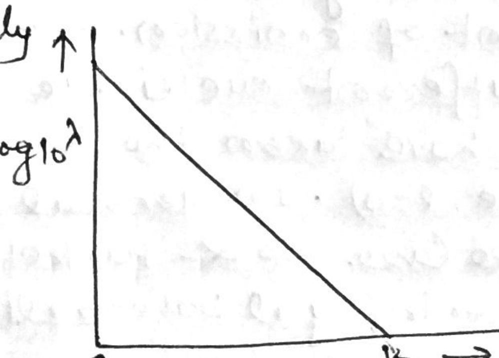
Where, E is in MeV, R is in unit of 10^{-15} and $z_D = (z-2)$ is the atomic No. of the residual Nucleus.

Taking the logrithm to the base 10, we write eqn (23), as

$$\log_{10}\lambda = \log_{10}\left(\frac{v_0}{2R}\right) + 1.28\, z_D^{1/2} R^{1/2} - 1.71\, z_D E^{-1/2} \quad \text{--- (24)}$$

eqn (24) is the Geiger-Nuttall law.

The agreement of theoretically result with experiment is remarkably good in case of even-even nuclei.



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Some remarks about alpha decay:-

(i) If the energy E of the α-particle and the decay constant λ are known for a given nucleus, r_0 can be calculated and the results are in excellet agreement with $\boxed{r = r_0 A^{1/3}}$ also the graph b/w $\log_{10}\lambda$ versus $1/\sqrt{E}$ is a st. line, which confirm Gamow's theory with experiment.

(2) The mean life $\tau = 1/\lambda$ is inversly proportional to the square root of α-particle energy.

(3) The existence of potential barrier explains that α-emission is not instantaneous. Also the probability tunnelling through the barrier is very small. However, α-particle within the nucleus present itself and again, till the conditions ~~at~~ of right penetration are not achieved.

(4). It is assumed that, α-particle pre-exists before it is ejected by the nucleus and α-particle is formed at the instant of emission, the probabilities being different for different nuclei due to odd-even effect.

(5) We have taken the α-emission from the ground state (i.e, $l=0$). In general the particle may be emitted with ($l \neq 0$) in those case the α-particle are subjected to the action of the centrifugal potential(V_l) in addition to the Coulomb potential (V_c) i.e

$$V = V_l + V_c = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + \frac{2(Z-2)e^2}{4\pi\epsilon_0 r}$$

and such the probability of α-emission is subsequently reduced.

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